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Analysis of fractional dynamical system and applications

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➢ Introduction

In recent decades, the field of epidemiology has witnessed a profound transformation, increasingly relying on mathematical modeling to comprehend the complexities of disease transmission dynamics within populations. The application of mathematical models in epidemiology has not only deepened our understanding of infectious disease spread but has also played a pivotal role in devising effective strategies for disease control and prevention.

The biological sciences have a research area called epidemiology that looks at all the factors that affect whether there are diseases and disorders. The outbreaks of various diseases in recent years, including SARS, Ebola, and COVID-19, have drawn more academics' attention to the study of epidemiology. Since these models aid in infection prevention and management, improving mathematical models of infectious epidemic diseases and discussions of their dynamics are extremely important in this regard. For a range of diseases, Numerous investigations on mathematical epidemic models have been conducted. These models can either be categorized as continuous-time [1–5], or models using discrete time. Continuous-time models are developed using differential equations, whereas discrete-time models are developed using difference equations. Since epidemiological information is only available at specific times, discrete models have recently been used widely instead of continuous models to assess and investigate infectious illnesses. Furthermore, compared to epidemic models that run continuously, epidemic models with discrete-time exhibit more intricate dynamics. Many notable discrete model types may be found in [6–9].

Epidemiology has made considerable use of fractional-order derivatives, having the bulk of these investigations concentrating on $SIR-type$ models [10–15]. While Naik examined the global dynamics of a memory-based, fractional-order SIR epidemic model in [16], Selvam et al. Researched using a discretization method to test the stability of the fractional-order SIR epidemic model in [17]. A unique fractional-order

SIR dengue model's analysis and numerical solution were looked into in [18]. An incommensurate order, delayed fractional-order SIR epidemic model was taken into consideration for stability analysis and bifurcation control in [19]. These studies are all grounded in continuous-time systems, which is important to note. Two common definitions of fractional derivatives are the Caputo derivative and the Riemann–Liouville derivative [20,21,22]. A useful tool for analyzing the properties of fractional-order systems, like fractional-order capacitors, is the Riemann–Liouville derivative. Conversely, the Caputo derivative finds extensive application in the mathematical modeling of systems and processes across diverse domains. The way the two derivatives describe the dynamic process is one of its main differences. It has been noted that when modeling and analyzing fractional-order capacitors in the time domain, the Riemann–Liouville derivative offers a more accurate option. Furthermore, numerical simulations of solutions to fractional ordinary differential equations frequently employ the Caputo derivative. In conclusion, although both derivatives have uses, the Caputo derivative is frequently employed in numerical simulations of fractional differential equations, whilst the Riemann–Liouville derivative is recommended for the analysis of fractional-order systems in the time domain.

□ History and Mathematical background of Dynamical System.

The beginning of mathematical modeling in epidemiology dates back to 1766, when Bernoulli developed a mathematical model to analyze the mortality of smallpox in England [23]. Bernoulli used his model to show that inoculation against the virus would increase the life expectancy at birth by about three years. A revision of the main findings and a presentation of the criticism by D'Alembert, appears recently in Dietz and Heesterbeek [24]. Lambert in 1772 as well as Laplace in 1812 extended the Bernoulli model by incorporating age-dependent parameters [25,26].

Many people regard French mathematician Henri Poincare as the founder of dynamical system. Poincare published two classical monographs, "New methods of Celestial Mechanics" (1892-1899) and "Lectures on Celestial Mechanics" (1905-1910). He successfully applied the results to the problem of the motion of three bodies and studied in detailed the behavior of solutions (frequency, stability, asymptotic). These papers included the Poincare recurrence theorem, which states that certain systems will, after a sufficiently long but finite time, return to a state very close to the initial

state.

Aleksandr Lyapunov developed many important approximation methods in 1899 which makeit possible to define the stability of sets of ordinary differential equation and also contributed the modern theory of the stability of a dynamical system.

However, further systematic research was absent until the beginning of the twentieth century with the pioneering work of Ross in 1911, which is considered the inaugural study of modern mathematical epidemiology [27]. Ross used a set of equations to approximate the discrete-time dynamics of malaria via a mosquito-based pathogen transmission [28]. Importantly, the past century has witnessed the rapid emergence and development of substantial theories in epidemics. In 1927, Kermack and McKendrick [29] derived the celebrated threshold theorem, which is one of the key results in epidemiology. It predicts – depending on the transmission potential of the infection – the critical fraction of susceptibles in the population that must be exceeded if an epidemic is to occur. Kermack and McKendrick published three seminal papers, establishing what is called the deterministic compartmental epidemic modelling [30–32], wherein they addressed the mass–action incident in disease transmission cycles, assuming that the probability of infection of a susceptible is analogous to the number of its contacts with infected individuals. This deterministic representation was in line with the Law of Mass Action [33] introduced by Guldberg and Waage in 1864 and renders the basic most commonly used SIR model, which assumes homogeneous mixing of the contacts and conservation of the total population and low rates of interaction. MacDonald extended Ross's model to explain in depth the transmission process of malaria. Utilizing modern computer power, the mathematical model for the dynamics and the control of mosquito transmitted pathogens provided robust results in real-word applications. Overall, the family of models they introduced is known by now as Ross–MacDonald models [34]. Moreover, the classic work of Bartlett [35] examined models and data to explore the factors that determine disease persistence in large populations. Arguably, a landmark book on mathematical modelling of epidemiological systems was published by Bailey [36] and highlighted the importance of public health decision making [37]. Given the diversity of infectious diseases studied since the middle of the 1950s, an impressive variety of epidemiological models have been developed.

In addition, we should highlight the 19th century works by Enko [38–40], who first published a probabilistic model for describing the epidemic of measles, yet in discrete time. This model is the precursor of the popular Reed-Frost chain binomial model introduced by Frost in 1928 in biostatistics' lectures at Johns Hopkins University [41]. It assumes that the infection spreads from an infected to a susceptible individual via a discrete time Markov chain, and set the basis of contemporary stochastic epidemic modelling, on which we will also focus in our present work.

In the late 20th century, Palestinian mechanical engineer Ali H. Nayfeh applied non-linear dynamics in mechanical and engineering systems. His pioneering work in applied nonlinear dynamics has been influential in the construction and maintenance of machines and structuresthat are common in daily life such as ships, cranes, bridges, buildings, jet engines, rocket engines, aircraft and spacecraft.

□ Recent works on the proposed topic

Moving to the 21st century, we mention some interesting works; Xing et al., [42] introduced a mathematical model on H7N9 influenza among migrant and resident birds, domestic poultry and humans in China. In this study they concluded that temperature seasonality might be a source of the disease, yet they suggested for the first time that controlling markets could help controlling outbreaks. Lee and Pietz [43] developed a mathematical model for Zika virus using logistic growth in human populations. Sun et al., [44] proposed a transmission model for cholera in China and observed that reducing the spread requires extensive immunization coverage of the population. Nishiura et al. [45] developed a Zika mathematical model which exhibited the same dynamics as dengue fever, and Khan et al. [46] introduced a model whereby a saturation function describes well the typhoid fever dynamics. Gui and young [47], developed a modified SIR model demonstrating nonlinearities in recovery rates. Their model exhibited a backward bifurcation phenomenon, which in turn implied that a plain reduction of the reproduction number less than one, was not rendered sufficient to stop the disease spread. Li et al. [48] constructed a multi-group brucellosis model and found out that the best way to contain the disease is to avoid cross infection of animal populations. Moreover, Yu and Lin [49] identified complex dynamical behaviour in epidemiological models and particularly the existence of multiple limit cycle

bifurcations using a predictor-prey model. Shi et al. [50] proposed an HIV model with a saturated reverse function to describe the dynamics of infected cells. Additionally, Bonyah et al. [51] developed a SIR model to study the dynamics of buruli ulcer and suggested policy measures to control the disease. Lastly, Zhang et al. [52] developed a model with a latent period of the disease wherein the person is not infectious with saturated incidence rates and treatment functions, called SEIR epidemic model. The performance of the various models is evaluated with the real dataset. More, the basic reproduction number of the COVID-19 virus propagation in the community is estimated based on the on-set phase dataset using multi-compartmental epidemiological model. Also, the maximum infection size, infection doubling time and the scope of the herd immunity are also inferred for COVID-19 in a population.

□ Scope of Study

The bridge between the mathematical sciences and biomedical sciences is heavily travelled. The correspondingly increased dialog between the disciplines has led to the establishment of the series**: Interdisciplinary Applied mathematics.** This will consist of monographs and high level texts from researchers working on the interplay between dynamical system and its biological applications.

□ Objectives of proposed research work

1. To explore the world of fractional dynamical systems by numerical approach.
2. To study the prerequisites: Calculus, Differential equations and Linear Algebra.
3. To study the concepts of state vectors and differentiate the dynamical world intothe discrete and the continuous.
4. To explore many instances of dynamical system in the real world, specially theexamples from biological background are to be discussed.
5. To discuss the biological problem.
6. To study the fixed points and stability of nonlinear systems.
7. To study the various population models and dynamics of infectious diseases.

□ Research Methodology

The following steps will be including in the methodology.

1. To refer the relevant reference books on dynamical system.

2. To refer the various articles/papers which are based on the proposed topic.
3. To develop the biological model using various numerical tools.
4. To learn the required mathematical software's like MATLAB and using the software's analyse the outcomes.
5. To find useful solution preferably quantitative.
6. To discuss biological interpretation of mathematical results in terms of insights and predictions.
7. To discuss a realistic dynamical representation of the important biological phenomena.
8. To discuss the global Ploblems of the topic.
9. To publish research paper in reputed journals.

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